Some Properties for Generalized Sakaguchi Type Functions of Complex Order

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Abstract

In this paper we shall study some properties of spiralike and Robertson functions of order α. Also, subordination results and integral means for generalized Sakaguchi type functions of complex order.

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1. Introduction

Let A denote the class of the functions of the form :

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and univalent in the open unit disc $U = \{z : |z| < 1\}$. Let $f \in A$ be given by (1.1) and $g(z)$ be given by

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \tag{1.2}$$

and let T denote the subclass of A whose elements can be expressed in the form:

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0) \tag{1.3}$$

Definition 1 (Hadamard product or convolution). Let a function f defined by (1.1) and g defined by (1.2), the Hadamard product (or convolution) $(f * g)$ is defined by

$$(f*g)(z)=z+\sum_{k=2}^{\infty}a_k b_k z^k=(g*f)(z).$$

Definition 2 [9]. A function $f(z)\in A$ is in $S^{\lambda}(\alpha)$, the class of λ -spirallike functions of order α ($0\le\alpha<1, |\lambda|<\frac{\pi}{2}$), if and only if

$$\operatorname{Re}\left\{e^{i\lambda}\frac{zf'(z)}{f(z)}\right\}>\alpha\cos\lambda \quad (z\in U). \tag{1.4}$$

We note that : $S^{\lambda}(0)=S^{\lambda}$, the class of λ -spirallike functions was introduced by Spacek [25] and $S^{0}(0)=S^{*}$ (see Silverman [19]). Further, a function $f(z)\in A$ is said to be in the class $C^{\lambda}(\alpha)$, the class of λ - Robertson functions of order α, if and only if

$$\operatorname{Re}\left\{e^{i\lambda}\left(1+\frac{zf''(z)}{f'(z)}\right)\right\}>\alpha\cos\lambda \quad (z\in U). \tag{1.5}$$

We note that $C^{\lambda}(0)=C^{\lambda}$, the class of functions $f(z)$, for which $zf'(z)$ is λ -spirallike in U introduced by Robertson [17] and the class $C^{\lambda}(\alpha)$ was introduced and studied by Chichra [3] and Sizuk [24] and $C^{0}(\alpha)=K(\alpha)$ were first introduced by Robertson [17].

We note that :

$$f(z)\in C^{\lambda}(\alpha)\Leftrightarrow zf'(z)\in S^{\lambda}(\alpha).$$

We denote by K the class of all convex functions (see Silverman [19]).

Also, Let Ω denote to the class of bounded analytic functions $\omega(z)$ in U and satisfying the conditions: $\omega(0)=0$ and $|\omega(z)|<|z|$ for $z\in U$.

Definition 3 [14] (Subordination Principle). For two functions $f(z)$ and $F(z)$, analytic in U, we say that $f(z)$ is subordinate to $F(z)$, written symbolically as follows:

$$f\prec F \text{ in } U \text{ or } f(z)\prec F(z) \quad (z\in U),$$

if there exists a Schwarz function $\omega(z)\in\Omega$, which (by definition) is analytic in U with

$$\omega(0)=0 \text{ and } |\omega(z)|<1 (z\in U)$$

such that

$$f(z)=F(\omega(z))\ (z\in U).$$

Indeed it is known that

$$f(z)\prec F(z)(z\in U)\Rightarrow f(0)=F(0)\text{ and } f(U)\subset F(U).$$

In particular, if the function $F(z)$ is univalent in U, we have the following equivalence

$$f(z)\prec F(z)(z\in U)\Leftrightarrow f(0)=F(0)\text{ and } f(U)\subset F(U).$$

An analytic function $f(z)\in A$ is said to be in the generalized Sakaguchi class $S(\alpha,s,t)$, if it satisfies

$$\operatorname{Re}\left\{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}\right\}>\alpha\ (z\in U) \tag{1.6}$$

for some $\alpha(0\le\alpha<1), s,t\in\mathbb{C}, |t|\le 1, s\ne t$ and for $z\in U$. The class $S(\alpha,s,t)$ defined and studied by Frasin [4] (see also [18], [11], [12]).

Remark 1. Putting $s=1$, we obtain $S(\alpha,1,t)=\Sigma(\alpha,t)$ studied by Owa et al. [15] and Goyal and Goswami [6].

For A,B fixed $-1\le B<A\le 1, s,t,\in\mathbb{C}, |t|\le 1,\ s\ne t$ and $b\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$, we define the subclass $S(s,t;b;A,B)$ of A consisting of functions f of the form (1.1) as follows:

$$1+\frac{1}{b}\left(\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-1\right)\prec\frac{1+Az}{1+Bz}. \tag{1.7}$$

From (1.7) and the definition of subordination, we obtain

$$1+\frac{1}{b}\left(\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-1\right)=\frac{1+A\omega(z)}{1+B\omega(z)}\quad(\omega(z)\in\Omega)$$

and hence we have

$$\left|\frac{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-1}{B\,\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-\left[B+(A-B)b\right]}\right|<1. \tag{1.8}$$

We let $T(s,t;b;A,B)=S(s,t;b;A,B)\cap T$, where T is defined by (1.3).

We note that for suitable choices of s,t,b,A and B, we obtain the following subclasses:

(i) Putting $s=b=1, t=0, A=1$ and $B=-1$, we have $S(1,0,1;1,-1)=S^*$.

(ii) Putting $s=b=1, t=0, A=1-2\alpha\left(0\le\alpha<1\right)$ and $B=-1$, we have

$S(1,0,1;1-2\alpha,-1)=S^*(\alpha)$ ،see Robertson [16]);

(iii) Putting $s=b=1, t=0, A=(1-2\alpha)\beta$ and $B=-\beta\left(0\le\alpha<1, 0<\beta\le 1\right)$, we have

$S(1,0,1;(1-2\alpha)\beta,-\beta)=S(\alpha,\beta)$ (see Gupta and Jain [7]);

(iv) Putting $s=1, t=0$ and $b=e^{-i\lambda}\cos\lambda(|\lambda|<\frac{\pi}{2})$, we have

$S(1,0,e^{-i\lambda}\cos\lambda;A,B)=S^{\lambda}(A,B)$ (see Aouf [1], with $\alpha=0$);

(v) Putting $s=1,\ t=0,\ b=e^{-i\lambda}\cos\lambda(|\lambda|<\frac{\pi}{2}), A=1$ and $B=-1,$ we have

$S(1,0,e^{-i\lambda}\cos\lambda;1,-1)=S^{\lambda}$ (see Spacek [25]);

(vi) Putting $s=1, t=0$, $b=e^{-i\lambda}\cos\lambda(|\lambda|<\frac{\pi}{2}), A=1-2\alpha(0\le\alpha<1)$ and $B=-1,$ we have

$S(1,0,e^{-i\lambda}\cos\lambda;1-2\alpha,-1)=S^{\lambda}(\alpha)$ (see Libera [9]);

(vii) Putting $s=1, t=0$, $b=e^{-i\lambda}\cos\lambda(|\lambda|<\frac{\pi}{2}), A=\left(1-2\alpha\right)\beta$ and

$B=-\beta\left(0\le\alpha<1, 0<\beta\le 1\right),$ we have

$S^{\lambda}(1,0,e^{-i\lambda}\cos\lambda;(1-2\alpha)\beta,-\beta)=S^{\lambda}(\alpha,\beta)$;

(viii) Putting $b=(1-\alpha)e^{-i\lambda}\cos\lambda(|\lambda|<\frac{\pi}{2}), A=1$ and $B=-1,$ we have

$S(s,t,(1-\alpha)e^{-i\lambda}\cos\lambda;1,-1)=\Re^{\lambda}(\alpha,s,t)$ (see Mathur et al. [13]).

(ix) Putting $b=e^{-i\lambda}\cos\lambda(|\lambda|<\frac{\pi}{2}),$ we have $S(s,t,e^{-i\lambda}\cos\lambda;A,B)=\Re^{\lambda}(s,t,A,B)$.

Definition 4 [27]. A sequence $\{c_k\}_{k=1}^{\infty}$ of complex numbers is said to be subordinating factor sequence if, whenever $f(z)$ of the form (1.1) is analytic, univalent and convex in U and

$$\sum_{k=1}^{\infty} a_k c_k z^k \prec f(z) \qquad (z\in U; a_1=1). \tag{1.9}$$

2. Main Results

First, we shall obtain some subordination results.

Unless otherwise mentioned we assume throughout the paper that $-1\le B<A\le 1, s, t, \in \mathbb{C}, |t|\le 1,\ \ s\ne t\ ,\ b\in\mathbb{C}^*$.

To prove our main results we need the following lemmas.

Lemma 1 [27]. The sequence $\{c_k\}_{k=1}^{\infty}$ is subordinating factor sequence if and only if

$$\operatorname{Re}\left\{1+2\sum_{k=1}^{\infty}c_k z^k\right\}>0 \qquad (z\in U).$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $S(s,t;b;A,B)$.

Lemma 2. A function $f(z)$ of the form (1.1) is in the class $S(s,t;b;A,B)$ if

$$\sum_{k=2}^{\infty}[|k-u_k|(1-B)+(A-B)|b||u_k|]|a_k|\le(A-B)|b|, \tag{2.1}$$

Where $\quad u_k=\sum_{j=1}^{k}s^{k-j}t^{j-1}.$

Proof. From (1.8), we have

$$\left|\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-1\right|-\left|B\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-\left[B+(A-B)b\right]\right|<0,$$

and

$$\frac{\sum_{k=2}^{\infty}|k-u_k||a_k|-B\sum_{k=2}^{\infty}|k-u_k||a_k|}{1-\sum_{k=2}^{\infty}|u_k||a_k|}\le(A-B)|b|$$

and hence

$$\sum_{k=2}^{\infty}[|k-u_k|(1-B)+(A-B)|b||u_k|]|a_k|\le(A-B)|b|.$$

Lemma 3. A necessary and sufficient condition for $f(z)\in T$ to be in the class $T(s,t;b;A,B)$ is that

$$\sum_{k=2}^{\infty}[|k-u_k|(1-B)+(A-B)|b||u_k|]|a_k|\le(A-B)|b|.$$

Proof. In view of Lemma 1, we need to prove the necessarily

$$\left|\frac{\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-1}{B\frac{(s-t)zf'(z)}{f(sz)-f(tz)}-\left[B+(A-B)b\right]}\right|$$

$$=\left|\frac{-\sum_{k=2}^{\infty}ku_ka_kz^k+\sum_{k=2}^{\infty}u_ka_kz^k}{Bz-B\sum_{k=2}^{\infty}ku_ka_kz^k-[B+(A-B)b](z-\sum_{k=2}^{\infty}u_ka_kz^k)}\right|$$

$$=\left|\frac{-\sum_{k=2}^{\infty}(k-u_k)a_kz^k}{-\left(A-B\right)bz+(A-B)b\sum_{k=2}^{\infty}u_ka_kz^k-B\sum_{k=2}^{\infty}(k-u_k)a_kz^k}\right|<1 \quad \left(z\in U\right).$$

Since $\mathrm{Re}(z)\le|z|$ for all z, we have

$$\mathrm{Re}\left\{\frac{\sum_{k=2}^{\infty}(k-u_k)a_kz^k}{\left(A-B\right)bz-(A-B)b\sum_{k=2}^{\infty}u_ka_kz^k+B\sum_{k=2}^{\infty}(k-u_k)a_kz^k}\right\}<1.$$

Choose values of z on the real axis so that $\frac{(s-t)zf'(z)}{f(sz)-f(tz)}$ is real. Then upon clearing the denominator in the last result and letting $z\to 1^-$ through real values, we have

$$\sum_{k=2}^{\infty}[|k-u_k|\left(1-B\right)+\left(A-B\right)|b||u_k|]|a_k|\le\left(A-B\right)|b|$$

This completes the proof of Lemma 3.

Remark 2. (i) Putting $b=\left(1-\alpha\right)e^{i\lambda}\cos\lambda\left(0\le\alpha<1,|\lambda|<\frac{\pi}{2}\right), A=1$ and $B=-1$ in Lemma 2, we obtain the result obtained by Mathur et al. [13, Theorem 2.3];

(ii) Putting $s=1,\ t=0,\ b=e^{i\lambda}\cos\lambda\left(|\lambda|<\frac{\pi}{2}\right), A=1$ and $B=-1$ in Lemma 2, we obtain the result obtained by Silverman [20] (see also Singh [23])

Let us denote by $S^*(s,t;b;A,B)$, the class of functions $f(z)\in A$ whose coefficients satisfy the inequality (2.1). We note that $S^*(s,t;b;A,B)\subset S(s,t;b;A,B)$.

Employing the technique used earlier by Attiya [2] and Srivastava and Attiya [26], we prove:

Theorem 1. Let $f(z)\in A$ satisfy the inequality (2.1).Then for every $h\in\mathbf{K}$, we have

$$\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}(f*h)(z)\prec h(z)\,(z\in U), \qquad (2.2)$$

and

$$\operatorname{Re}(f(z))>-\frac{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|}{|2-s-t|(1-B)+(A-B)|b||s+t|}, \qquad (z\in U). \qquad (2.3)$$

The constant factor $\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}$ in the subordination result (2.2) is the best possible.

Proof. Let $f(z)\in S^*(s,t;b;A,B)$ and let $h(z)=z+\sum_{k=2}^{\infty}d_k z^k\in k$. Then we have

$$\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}(f*h)(z)=$$

$$\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}\left(z+\sum_{k=2}^{\infty}a_k d_k z^k\right). \qquad (2.4)$$

Thus, by Definition 4, the subordination result (2.2) will hold true if the sequence

$$\left\{\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}a_k\right\}_{k=1}^{\infty} \qquad (2.5)$$

is a subordinating factor sequence, with $a_1=1$. In view of Lemma 1, this is equivalence to the following inequality:

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|}a_k z^k\right\}>0\,(z\in U). \qquad (2.6)$$

Now, since

$$\Psi(k)=|k-u_k|(1-B)+(A-B)|b||u_k|$$

is an increasing function of $k\ (k\geq 2)$, we have

$$\operatorname{Re}\left\{1+\sum_{k=1}^{\infty}\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|}a_k z^k\right\}$$

$$= \mathrm{Re}\left\{1+\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|}z\right.$$

$$\left.+\frac{1}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|}\sum_{k=2}^{\infty}\left[|2-s-t|(1-B)+(A-B)|b||s+t|\right]a_k z^k\right\}$$

$$\geq 1-\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|}r$$

$$-\frac{1}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|}\sum_{k=2}^{\infty}\left[|2-s-t|(1-B)+(A-B)|b||s+t|\right]|a_k|r^k$$

$$>1-\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|}r-\frac{(A-B)|b|}{|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|}r$$

$$>0 \qquad (|z|=r<1),$$

where we have also made use of assertion (2.1) of Lemma 2. Thus (2.6) holds true in U. This proves the inequality (2.2). The inequality (2.3) follows from (2.2) by taking the convex function $h(z)=\frac{z}{1-z}=z+\sum_{k=2}^{\infty}z^k$.

To prove the sharpness of the constant $\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}$, we consider the function $f_0(z)\in S^*(s,t;b;A,B)$ is given by

$$f_0(z)=z-\frac{(A-B)|b|}{|2-s-t|(1-B)+(A-B)|b||s+t|}z^2. \tag{2.7}$$

Thus from the relation (2.2), we obtain

$$\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}(f_0)(z)\prec\frac{z}{1-z}\ (z\in U). \tag{2.8}$$

It can easily be verified that

$$\min_{|z|\leq 1}\mathrm{Re}\left\{\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}(f_0)(z)\right\}=-\frac{1}{2}. \tag{2.9}$$

This shows that the constant $\frac{|2-s-t|(1-B)+(A-B)|b||s+t|}{2[|2-s-t|(1-B)+(A-B)|b||s+t|+(A-B)|b|]}$ is the best possible. This completes the proof of Theorem 1.

Remark 3. (i) Putting $b=(1-\alpha)e^{i\lambda}\cos\lambda\left(0\leq\alpha<1,|\lambda|<\frac{\pi}{2}\right), A=1$ and $B=-1$ in Theorem 1, we obtain the result obtained by Mathur et al. [13, Theorem 3.1];

(ii) Putting $s=1, t=0,\ b=e^{i\lambda}\cos\lambda\left(|\lambda|<\frac{\pi}{2}\right),\ A=1$ and $B=-1$ in Theorem 1, we obtain the result obtained by Singh [23, Theorem 2.1];

(iii) Putting $s=1, t=0, b=(1-\alpha)e^{i\lambda}\cos\lambda\left(0\le\alpha<1, |\lambda|<\frac{\pi}{2}\right), A=1$ and $B=-1$ in Theorem 1, we obtain the result obtained by Kwon and Owa [8, Theorem 2.4];

(iv) Putting $s=b=1, t=0, A=1$ and $B=-1$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.4];

(v) Putting $s=b=1, t=0, A=1-2\alpha\left(0\le\alpha<1\right)$ and $B=-1$ in Theorem 1, we obtain the result obtained by Frasin [5, Corollary 2.3].

Also, we establish subordination results for the associated subclasses,

$$S^{*}(\alpha,\beta),\ S^{*\lambda}(A,B),\ S^{*\lambda}(\alpha,\beta),\ \Re^{*\lambda}(s,t,A,B)\ .$$

Putting $s=b=1, t=0, A=(1-2\alpha)\beta$ and $B=-\beta\left(0\le\alpha<1, 0<\beta\le1\right)$ in Theorem 1, we obtain the following corollary

Corollary 1. Let the function $f(z)$ defined by (1.1) be in the class $S^{*}(\alpha,\beta)$ and suppose that $h(z)\in\mathbf{K}$. Then

$$\frac{1+\beta(3-2\alpha)}{2[1+\beta(5-4\alpha)]}(f*h)(z)\prec h(z)\,(z\in U), \qquad (2.10)$$

and

$$\mathrm{Re}\left(f(z)\right)>-\frac{1+\beta(5-4\alpha)}{1+\beta(3-2\alpha)}, \qquad (z\in U).$$

The constant factor $\frac{1+\beta(3-2\alpha)}{2[1+\beta(5-4\alpha)]}$ in the subordination result (2.10) is the best possible.

Putting $s=1,\ t=0$ and $b=e^{-i\lambda}\cos\lambda$ in Theorem 1, we obtain the following corollary.

Corollary 2. Let the function $f(z)$ defined by (1.1), be in the class $S^{*\lambda}(A,B)$ and suppose that $h(z)\in\mathbf{K}$. Then

$$\frac{(A-B)+(1-B)\sec\lambda}{2[2(A-B)+(1-B)\sec\lambda]}(f*h)(z)\prec h(z)\,(z\in U), \qquad (2.11)$$

and

$$\mathrm{Re}\left(f(z)\right)>-\frac{2(A-B)+(1-B)\sec\lambda}{(A-B)+(1-B)\sec\lambda}, \qquad (z\in U).$$

The constant factor $\frac{(A-B)+(1-B)\sec\lambda}{2[2(A-B)+(1-B)\sec\lambda]}$ in the subordination result (2.11) is the best possible.

Putting $s=1, t=0$, $b=e^{-i\lambda}\cos\lambda, A=(1-2\alpha)\beta$ and $B=-\beta$ $(0\le\alpha<1, 0<\beta\le 1)$, in Theorem 1, we obtain the following corollary.

Corollary 3. Let the function $f(z)$ defined by (1.1) be in the class $S^{*\lambda}(\alpha,\beta)$ and suppose that $h(z)\in\mathbf{K}$. Then

$$\frac{2\beta(1-\alpha)+(1+\beta)\sec\lambda}{2[4\beta(1-\alpha)+(1+\beta)\sec\lambda]}(f*h)(z)\prec h(z)(z\in U), \tag{2.12}$$

and

$$\operatorname{Re}(f(z))>-\frac{4\beta(1-\alpha)+(1+\beta)\sec\lambda}{2\beta(1-\alpha)+(1+\beta)\sec\lambda}, \qquad (z\in U).$$

The constant factor $\frac{2\beta(1-\alpha)+(1+\beta)\sec\lambda}{2[4\beta(1-\alpha)+(1+\beta)\sec\lambda]}$ in the subordination result (2.12) is the best possible.

Putting $b=e^{-i\lambda}\cos\lambda, A=1$ and $B=-1$ in Theorem 1, we obtain the following corollary.

Corollary 4. Let the function $f(z)$ defined by (1.1) be in the class $\Re^{*\lambda}(s,t,A,B)$ and suppose that $h(z)\in\mathbf{K}$. Then

$$\frac{|2-s-t|(1-B)+(A-B)\cos\lambda|s+t|}{2[|2-s-t|(1-B)+(A-B)\cos\lambda|s+t|+(A-B)\cos\lambda]}(f*h)(z)\prec h(z)(z\in U), \tag{1.13}$$

and

$$\operatorname{Re}(f(z))>-\frac{|2-s-t|(1-B)+(A-B)\cos\lambda|s+t|+(A-B)\cos\lambda}{|2-s-t|(1-B)+(A-B)\cos\lambda|s+t|}, \quad (z\in U).$$

The constant factor $\frac{|2-s-t|(1-B)+(A-B)\cos\lambda|s+t|}{2[|2-s-t|(1-B)+(A-B)\cos\lambda|s+t|+(A-B)\cos\lambda]}$ in the subordination result (2.13) is the best possible.

Next, we shall obtain some integral mean results.

In [19], Silverman found that the function $f_2(z)=z-\frac{z^2}{2}$ is often extremal over the family T. He applied this function to resolve his integral means inequality, conjectured in [21] and settled in [22], that

$$\int_0^{2\pi}\left|f(re^{i\varphi})\right|^{\delta}d\varphi\le\int_0^{2\pi}\left|f_2(re^{i\varphi})\right|^{\delta}d\varphi,$$

for all $f(z) \in T$, $\delta > 0$ and $0 < r < 1$. In [22], Silverman proved his conjecture for the classes $T^*(\alpha)$, where $T^*(\alpha) = S^*(\alpha) \cap T$.

In 1925, Littlewood [10] proved the following subordination theorem.

Lemma 4 [10]. If the functions $f(z)$ and $h(z)$ are analytic in U with $f(z) \prec h(z)$, then for $\delta > 0$ and $z = re^{i\theta} (0 < r < 1)$, we have

$$\int_0^{2\pi} \left| f(re^{i\theta}) \right|^{\delta} d\theta \le \int_0^{2\pi} \left| h(re^{i\theta}) \right|^{\delta} d\theta.$$

Theorem 2. If $f(z) \in T(s,t,b,A,B)$ and $\delta > 0$, $0 < |z| = r < 1$, then for function $h(z) \in \mathbf{K}$, we have

$$\frac{\varphi(2)}{\varphi(2) + (A-B)|b|} \int_0^{2\pi} \left| (f * h)(re^{i\theta}) \right|^{\delta} d\theta \le 2 \int_0^{2\pi} \left| h(re^{i\theta}) \right|^{\delta} d\theta, \tag{2.14}$$

where $\varphi(2) = |2 - s - t| (1 - B) + (A - B)|b||s + t|$.

Proof. The proof follows from Theorem 1 and Lemma 4.

Putting $s = b = 1$, $t = 0$, $A = 1$ and $B = -1$ in Theorem 2, we obtain the following corollary.

Corollary 5. Let the function $f(z)$ defined by (1.3) be in the class S^* and suppose that $h \in \mathbf{K}$. Then

$$\frac{1}{3} \int_0^{2\pi} \left| (f * h)(re^{i\theta}) \right|^{\delta} d\theta \le \int_0^{2\pi} \left| h(re^{i\theta}) \right|^{\delta} d\theta, \tag{2.15}$$

Putting $s = b = 1$, $t = 0$, $A = 1 - 2\alpha \, (0 \le \alpha < 1)$ and $B = -1$ in Theorem 2, we obtain the following corollary.

Corollary 6. Let the function $f(z)$ defined by (1.3) be in the class $S^*(\alpha)$ and suppose that $h \in \mathbf{K}$. Then

$$\frac{2 - \alpha}{2(3 - 2\alpha)} \int_0^{2\pi} \left| (f * h)(re^{i\theta}) \right|^{\delta} d\theta \le \int_0^{2\pi} \left| h(re^{i\theta}) \right|^{\delta} d\theta, \tag{2.16}$$

Putting $s = b = 1$, $t = 0$, $A = (1 - 2\alpha)\beta$ and $B = -\beta \, (0 \le \alpha < 1, 0 < \beta \le 1)$ in Theorem 2, we obtain the following corollary.

Corollary 7. Let the function $f(z)$ defined by (1.3) be in the class $S^*(\alpha, \beta)$ and suppose that $h \in \mathbf{K}$. Then

$$\frac{1 + \beta(3 - 2\alpha)}{2[1 + \beta(5 - 4\alpha)]} \int_0^{2\pi} \left| (f * h)(re^{i\theta}) \right|^{\delta} d\theta \le \int_0^{2\pi} \left| h(re^{i\theta}) \right|^{\delta} d\theta, \tag{2.17}$$

Putting $s=1, t=0$ and $b=e^{-i\lambda}\cos\lambda\left(|\lambda|<\frac{\pi}{2}\right)$ in Theorem 2, we obtain the following corollary.

Corollary 8. Let the function $f(z)$ defined by (1.3) be in the class $S^{*\lambda}(A,B)$ and suppose that $h\in\mathbf{K}$.Then

$$\frac{(A-B)+(1-B)\sec\lambda}{2(A-B)+(1-B)\sec\lambda}\int_0^{2\pi}\left|(f*h)(re^{i\theta})\right|^{\delta}d\theta\le 2\int_0^{2\pi}\left|h(re^{i\theta})\right|^{\delta}d\theta, \tag{2.18}$$

Putting $s=1, t=0$, $b=e^{-i\lambda}\cos\lambda\left(|\lambda|<\frac{\pi}{2}\right), A=(1-2\alpha)\beta$ and $B=-\beta\,(0\le\alpha<1, 0<\beta\le1)$ in Theorem 2, we obtain the following corollary.

Corollary 9. Let the function $f(z)$ defined by (1.3) be in the class $S^{*\lambda}(\alpha,\beta)$ and suppose that $h\in\mathbf{K}$.Then

$$\frac{2\beta(1-\alpha)+(1+\beta)\sec\lambda}{2[4\beta(1-\alpha)+(1+\beta)\sec\lambda]}\int_0^{2\pi}\left|(f*h)(re^{i\theta})\right|^{\delta}d\theta\le\int_0^{2\pi}\left|h(re^{i\theta})\right|^{\delta}d\theta, \tag{2.19}$$

Putting $b=e^{-i\lambda}\cos\lambda\left(|\lambda|<\frac{\pi}{2}\right), A=1$ and $B=-1$ in Theorem 2, we obtain the following corollary.

Corollary 10. Let the function $f(z)$ defined by (1.3) be in the class $\Re^{*\lambda}(s,t,A,B)$ and suppose that $h\in\mathbf{K}$.Then

$$\frac{|2-s-t|(1-B)+(A-B)\cos\lambda|s+t|}{2[|2-s-t|(1-B)+(A-B)\cos\lambda|s+t|+(A-B)\cos\lambda]}\int_0^{2\pi}\left|(f*h)(re^{i\theta})\right|^{\delta}d\theta\le\int_0^{2\pi}\left|h(re^{i\theta})\right|^{\delta}d\theta, \tag{2.20}$$

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