ANTISYMMETRIC ASYMPTOTIC EXPANSIONS OF THE DISPERSION RELATION FOR A COMPRESSIBLE LAMINATED PLATE

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ABSTRACT

The dispersion relation associated with the propagation of waves in a pre-stressed 4-ply laminated plate is derived and analyzed, both numerically and asymptotically. Each layer is assumed to be composed of a linear isotropic elastic material. Numerical solutions of the relation are first presented. After presentation of these numerical solutions, particular focus is applied to the short wave regime, within which appropriate asymptotic approximations are established. These are shown to provide excellent agreement with the numerical solution over a surprisingly larger than might be expected wave number regime. It is envisaged that these solutions might offer some potential for estimation of truncation error for wave number integrals and thereby enable the development of hybrid numerical-asymptotic methods to determine transient structural response to impact.

1. INTRODUCTION

Laminated structures are used frequently within components of modern engineering structures. One of the most popular areas of their use within modern engineering science is within the aerospace industry. One particularly important application of laminated structures is to provide a high strength to weight ratio. This is becoming an increasingly important factor as manufacturers try to design lighter and more fuel efficient aircraft to reduce their carbon footprint. In addition to use of layered structures within modern engineering components, there are many examples of layered structures found to be naturally occurring. In this present paper we wish to investigate the dispersion of waves in a 3-layer symmetric plate. Each layer
is assumed to be composed of a linear isotropic elastic solid. The analysis of dispersion in a single isotropic elastic layer has a long history. A detailed investigation, together with the systematic development of long wave models, can be found in the extensive research monograph [1]. We also remark that within this excellent monograph a detailed historical reference list may be found. For a single layer these ideas have been extended to pre-stressed and anisotropic media by a number of authors, see for example [2-4] and references therein. Most of previous asymptotic studies have focused on long wave theories. One paper that develops a short wave theory is [5].

This paper is organized as follows. In Section 2 the necessary basic elasticity equations are briefly reviewed and applied to 2-dimensional motion appropriate for a layer. Within Section 3 the dispersion relation associated with wave propagation in the 3-layer plate is derived. In Section 4, two numerical illustrations of the dispersion relation are presented. In Section 5, a short wave asymptotic analysis of the dispersion relation is carried out. This confirms that the fundamental modes may approach either the associated surface or interfacial wave speed. For the harmonics, analytical approximations, giving phase speed explicitly for each harmonic as a function of material parameters and mode number, are established.

2. BASIC EQUATIONS

We shall consider the propagation of travelling waves in a Pre-stressed symmetric 4-ply laminate. The laminate is formed of two identically outer layers of height h bonded perfectly to an inner core of height (2d) and is symmetrical about its mid plane and of infinite extent in the remaining
spatial directions. An appropriate Cartesian coordinate system is chosen coincident with the axes of \( \{ B \} \) such that \((Ox_2)\) is normal to the plane of the laminate and the origin lies on the mid plane of the structure. In order to further simplify subsequent numerical and analytical investigation a state of plane strain is assumed such that both \((u_1)\) and \((u_2)\) are independent of \((x_3)\) and \((u_3)\). The two non-trivial components of the equations of motion take the form

\[
\begin{align*}
A_1 u_{1,1} + (A_1 + A_2) u_{2,2} + A_2 u_{1,2} &= \rho \ddot{u}_1, \quad (2.1) \\
(A_1 + A_2) u_{1,1} + A_1 u_{2,1} + A_2 u_{2,2} &= \rho \ddot{u}_2. \quad (2.2)
\end{align*}
\]

We now seek solutions of \((2.1)\) and \((2.2)\) yielding two linear homogeneous equations with non-trivial solutions if

\[
\alpha_2 \gamma_2 q^4 + \{ \alpha_2 (\rho v^2 - \alpha_1) + \gamma_2 (\rho v^2 - \gamma_1) + \beta^2 \} q^2 + (\rho v^2 - \alpha_1) (\rho v^2 - \gamma_1) = 0. \quad (2.3)
\]

With

\[
\alpha_i = A_{ii}, \quad i, i \in \{1, 2\}, \quad \gamma_1 = A_1, \quad \gamma_2 = A_2, \quad \beta = \alpha_1 + \gamma_2 - \sigma_2, \quad (2.4)
\]

Where the fact that \(\alpha_1 + \alpha_2\) has been used, \(\sigma_2\) is the principal Cauchy stress normal to the layer in \(B_\varepsilon\) this being related to the material parameters through the relation \(\sigma_2 = \gamma_2 - A_1\).

If the two roots of equation \((2.3)\) are denoted by \(q_1^2\) and \(q_2^2\) where it is noted that \(q_1\) and \(q_2\) may be either real, purely imaginary or complex conjugates, we note for future reference that
\[ q_1^2 + q_2^2 = \frac{\alpha_2}{\alpha_2} (\mu_e \nu^2 - \alpha_1 \gamma_2 (\mu_e \nu^2 - \gamma_1) - \nu^2), \quad \beta_2^2 = \frac{(\mu_e \nu^2 - \alpha_1 \gamma_2 (\mu_e \nu^2 - \gamma_1))}{\alpha_1 \gamma_2}. \quad (2.5) \]

The solutions for \( U \) and \( V \) may be presented as a linear combinations of the four generally independent solutions associated with the roots of (2.3). On making use of the equations of motion (2.1), these solutions may be recast in the form

\[ U = \sum_{m=1}^{2} \left( U^{(2m-1)} e^{k \beta m x_2} + U^{(2m)} e^{-k \beta m x_2} \right) \quad (2.6) \]

\[ V = \sum_{m=1}^{2} \frac{g(q_m)}{\beta} q_m \left( U^{(2m-1)} e^{k \beta m x_2} - U^{(2m)} e^{-k \beta m x_2} \right), \quad (2.7) \]

Where

\[ g(q_m) = \alpha_1 \gamma_2 (\mu_e \nu^2 - \gamma_1) - \nu^2 \quad (2.8) \]

and \( U^{(i)}, \quad i \in \{1,2,3,4\} \) are disposable constants. On making use of the equations (2.6)-(2.7) with equation (2.2), and noting that the unit normal to the upper surface of the layer is of the form given by \( n = (0,1,0) \), solutions for the linearised incremental traction components may now be obtained, thus

\[ \tau_1 = \sum_{m=1}^{2} \frac{f(q_m)}{\beta} q_m \left( U^{(2m-1)} e^{k \beta m x_2} - U^{(2m)} e^{-k \beta m x_2} \right), \quad (2.9) \]

\[ \tau_2 = \sum_{m=1}^{2} \frac{h(q_m)}{\beta} q_m \left( U^{(2m-1)} e^{k \beta m x_2} + U^{(2m)} e^{-k \beta m x_2} \right), \quad (2.10) \]

Where

\[ f(q_m) = \beta \gamma_2 q_m^2 + (\gamma_2 - \alpha_2) g(q_m), \quad (2.11) \]

\[ h(q_m) = \alpha_2 g(q_m) - \alpha_1 \beta. \quad (2.12) \]
3. DERIVATION OF THE DISPERSION RELATION

In this section we derive the dispersion relation associated with wave propagation in our symmetric four-ply laminated plate. Such a relation provides an implicit relationship between phase speed and wave number. The boundary and continuity conditions are expressed explicitly as

\[ \tau_1 = \tau_2 = 0 \text{ at } x_2 = \pm (d + h), \]

\[ U_1 = \overline{U}_1, \quad U_2 = \overline{U}_2, \quad \tau_1 = \overline{\tau}_1, \quad \tau_2 = \overline{\tau}_2, \text{ at } x_2 = \pm d. \]  

(3.1)

Inserting the appropriate forms of the governing equations (2.6)-(2.12) into equation (3.1) will give rise to a system of twelve homogeneous equations in twelve unknowns. However, this system may be reduced to two systems of six equations in six unknowns. We shall do this by appropriate selection of conditions at the mid-plane. For antisymmetric solutions \( \overline{\gamma} \) and \( \overline{\varphi}_1 \) vanish on the mid-plane, implying from equations (2.7) and (2.10) that \( \overline{U}_1 = - \overline{U}_2 \) thus equations (3.1) reduce to

\[ \sum_{m=1}^{2} \frac{f(q_m)}{\beta} q_m (U^{(2m-1)} e^{k\mu m (d+h)} - U^{(2m)} e^{-k\mu m (d+h)}) = 0, \]

\[ \sum_{m=1}^{2} \frac{h(q_m)}{\beta} (U^{(2m-1)} e^{k\mu m (d+h)} + U^{(2m)} e^{-k\mu m (d+h)}) = 0, \]

\[ \sum_{m=1}^{2} U^{(2m-1)} e^{k\mu m d} + U^{(2m)} e^{-k\mu m d} - \overline{\gamma}^{(2m-1)} \overline{\varepsilon}_m = 0, \]  

(3.2)

\[ \sum_{m=1}^{2} \frac{g(q_m)}{\beta q_m} (U^{(2m-1)} e^{k\mu m d} - U^{(2m)} e^{-k\mu m d}) \overline{\gamma}^{(2m-1)} \overline{\varepsilon}_m = 0, \]  

\[ \sum_{m=1}^{2} \frac{g(q_m)}{\beta q_m} (U^{(2m-1)} e^{k\mu m d} - U^{(2m)} e^{-k\mu m d}) \overline{\gamma}^{(2m-1)} \overline{\varepsilon}_m = 0, \]  

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\[
\sum_{m=1}^{2} \frac{f(q_m)}{\beta q_m} \left( e^{k_\mu m \alpha} - e^{-k_\mu m \alpha} \right) - \frac{f(p_m)}{\beta p_m} U^{(2m-1)} s_m = 0,
\]
\[
\sum_{m=1}^{2} \frac{h(q_m)}{\beta} \left( U^{(2m-1)} e^{k_\mu m (\alpha + n)} + U^{(2m)} e^{-k_\mu m (\alpha + n)} \right) - \frac{h(p_m)}{\beta} U^{(2m-1)} \ell_m = 0,
\]

Equations (3.2) will yield non-trivial solutions provided

\[
\begin{vmatrix}
q_1 h(q_1) c_1 & q_1 h(q_1) s_1 & q_2 h(q_2) c_2 & q_2 h(q_2) s_2 & 0 & 0 \\
f(q_1) s_1 & f(q_1) c_1 & f(q_2) c_2 & f(q_2) s_2 & 0 & 0 \\
q_1 e_1 & 0 & q_2 e_2 & 0 & p_1 \beta \ell_1 & p_2 \beta \ell_2 \\
0 & g(q_1) & 0 & g(q_2) & g(p_1) \ell_1 & g(p_2) \ell_2 \\
0 & f(q_1) & 0 & f(q_2) & f(p_1) \ell_1 & f(p_2) \ell_2 \\
q_1 h(q_1) & 0 & q_1 h(q_2) & 0 & p_1 \hat{\ell}(p_1) \ell_1 & p_2 \hat{\ell}(p_2) \ell_2 \\
\end{vmatrix} = 0.
\]

(3.3)

An explicit representation of the dispersion relation is obtained as

\[
q_1 q_2 f(q_1) h(q_1) \delta_1 - \{ q_1 f(q_2) h(q_1) c_1 s_1 - q_2 f(q_1) h(q_2) c_2 s_1 \} \delta_2 + \{ q_1 f(q_2) h(q_1) c_1 c_2 - q_2 f(q_1) h(q_2) c_2 c_1 \} \delta_3
\]
\[
- \{ q_1 f(q_2) h(q_1) c_2 s_1 - q_2 f(q_1) h(q_2) c_1 s_1 \} \delta_5 + q_1 q_2 f(q_2) h(q_2) \delta_6 = 0,
\]

(3.4)

Within which

\[
\delta_1 = \hat{\ell}_1 \hat{\ell}_2 p_1 \{ (f(q_2) g(p_2) - f(p_2) g(q_2))(\beta \hat{\ell}(p_1) - \beta h(q_2)) \} - \hat{\ell}_2 \hat{\ell}_1 p_2 \{ (f(q_2) g(p_1) - f(p_1) g(q_2))(\beta \hat{\ell}(p_2) - \beta h(q_2)) \},
\]

\[
\delta_2 = \hat{\ell}_1 \hat{\ell}_2 p_1 p_2 \beta \{ (f(q_1) g(q_2) - f(q_2) g(q_1))(\hat{\ell}(p_1) - \hat{\ell}(p_2)) \},
\]

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\[ \delta_2 = \lambda^2_c p_1 q_2 \left( f(q_1) g(p_2) - \hat{f}(p_2) g(q_1) \right) (\beta h(p_1) - \beta \hat{h}(q_2)) + \lambda^2_2 \lambda_1 p_2 q_2 \left( f(q_1) g(p_1) - \hat{f}(p_1) g(q_1) \right) (\beta h(p_2) - \beta \hat{h}(q_2)). \]

\[ \delta_4 = \lambda^2_1 \lambda_2 q_2 q_1 \left( f(q_2) g(p_2) - \hat{f}(p_2) g(q_2) \right) (\beta h(p_1) - \beta \hat{h}(q_1)) - \lambda^2_2 \lambda_1 p_2 q_1 \left( f(q_1) g(p_1) - \hat{f}(p_1) g(q_1) \right) (\beta h(p_2) - \beta \hat{h}(q_1)). \]

\[ \delta_5 = \lambda^2_1 \lambda_2 q_2 q_1 \beta \left( f(p_1) g(p_2) - \hat{f}(p_2) g(p_1) \right) (h(q_1) - h(q_2)). \]

\[ \delta_6 = \lambda^2_1 \lambda_2 p_1 \left( f(q_1) g(p_2) - \hat{f}(p_2) g(q_1) \right) (\beta h(p_1) - \beta \hat{h}(q_1)) - \lambda^2_2 \lambda_1 p_2 \left( f(q_1) g(p_1) - \hat{f}(p_1) g(q_1) \right) (\beta h(p_2) - \beta \hat{h}(q_1)). \]

### 4. NUMERICAL ANALYSIS

We first use the modified Varga strain energy function energy function in the form

\[ W = (\lambda_1 + \lambda_2 + \lambda_3 - 3) + \frac{1}{2} \kappa (f - 1)^2, \quad J = \det F = \lambda_1 \lambda_2 \lambda_3, \]

where \( \kappa \) is a shear modulus and \( \kappa \) the bulk modulus. Figure 1 presents a plot of scaled phase speed against scaled wave number is illustrated for the first twenty eight branches of the dispersion relation (3.4). In the high wave number region (kh, kd \( \rightarrow \infty \)) the fundamental mode tends to a distinct surface wave speed, whilst all harmonics asymptote from above to a longitudinal wave speed associated with the two outer layers \( v_1 \approx 0.946 \). In this case it is again observed numerically that one of \( q_1 \) or \( q_2 \) is imaginary, the other real, whilst \( p_1 \) and \( p_2 \) are real, as designed in case 1.
Fig 1: $\tilde{q}$ against $k$ for a laminate formed of layers of materials with strain energy (4.1): outer layers $\mu = 2.0, k = 1.6, \lambda_1 = 1.6, \lambda_2 = 0.8, \lambda_3 = 0.7$; inner core $\tilde{\mu} = 2.5, \tilde{k} = 1.5, \tilde{\lambda}_1 = 1.9, \tilde{\lambda}_2 = 0.9, \tilde{\lambda}_3 = 0.4$; with $\nu_1 = 0.9, \nu_2 = 1.5, \tilde{\nu}_1 = 1.1, \tilde{\nu}_2 = 1.5, \tilde{\nu}_3 = 0.9, \tilde{\nu}_R = 0.6$.

Case (1): $q_1 = i\tilde{q}_1, q_2$ real, $p_1$ and $p_2$ real or complex conjugates, as $kh \to \infty q_1 \to 0$, $\rho \nu^2 = \rho \nu_1 = \alpha_1$.

The flattening of the dispersion curves occurs to form wave fronts around the longitudinal wave speed associated with the inner core $\tilde{v}_1 \approx 1.16$ and the shear wave speed associated with the outer layers $\tilde{v}_2 \approx 1.543$. Figure (4.1) affords an example in which $v_3 = \min (v_1, v_2, v_3)$ but is not the limiting wave speed of the harmonics. A second case analogous to case 1, occurs
when all harmonics tend to a longitudinal wave associated with the outer layers. This case is classified through,

Case (2): \( \nu_1 = i \bar{\nu}_1 , q_2 \) real, \( \nu_1 \) and \( q_2 \) real or complex conjugates,

as \( k h \rightarrow \infty \), \( \nu_1 \rightarrow 0 \), \( \nu_1^2 \rightarrow \bar{\nu}_1^2 = \bar{\alpha}_1 \).

The second form of strain energy employed is the modified Neo-Hookean

\[
W = \nu_1 (l - 3) + \frac{1}{2} \kappa (j - 1)^2
\]

(4.2)

Fig 2: \( \bar{\nu} \) against \( k \) for a laminate formed of layers of materials with strain energy (4.2): outer layers \( \nu_1 = 0.2 , k = 0.3 , \lambda_1 = 1.4 , \lambda_2 = 1.2 , \lambda_3 = 1.0 \); inner core \( \nu_1 = 1.4 , \lambda_1 = 1.7 , \lambda_2 = 0.4 , \lambda_3 = 2.0 \); with \( \nu_1 = 0.9 , \nu_2 = 1.6 , \nu_1 = 2.4 , \nu_2 = 2.3 , \nu_2 = 0.3 \); \( \nu_2 \) = 0.5.
with \( c_1 \) a material parameters and \( k \) is the bulk modulus. In figure (2) we again note that in the low wave number region \((kh, k \to 0)\) only the fundamental mode retains a finite wave speed, with all harmonics having an associated large wave speed. The flattening of the dispersion curves to form wave fronts around \( v_1 \approx 0.98 \), the longitudinal wave speed associated with the two outer layers, and around a shear wave speed in the inner core, \( \bar{v}_2 \approx 2.31 \).

Case (3): \( q_1 = i\bar{q}_1 q_2 \) real, \( p_1 \) and \( v_2 \) real or complex conjugates, as \( kh \to \infty \), \( \bar{q}_1 \to 0 \) a \( \bar{v}^2 \to \bar{v}_1^2 = \gamma_1 \)

We again see the flattening of the harmonics around the longitudinal wave speed associated with the inner core \( \bar{v}_1 \approx 0.96 \) and around the shear wave speed associated the outer layers \( v_2 \approx 1.75 \).

Case (4): \( p_1 = i\bar{p}_1 q_2 \) real, \( q_1 \) and \( q_2 \) real or complex conjugates, as \( kh \to \infty \), \( \bar{p}_1 \to 0 \) a \( \bar{v}^2 \to \bar{v}_2^2 = \bar{v}_1 \)

The final strain energy function employed is the Blatz-Ko strain energy

\[
W = \frac{\mu}{2} \left( \frac{L^2}{J^2} + 2J - 5 \right), \tag{4.3}
\]

In figure (3) a plot for the twenty eight branches of the dispersion relation (3.4) is presented for layers with Blatz-Ko strain energy (4.3). In the high
wave number limit, the fundamental mode and all harmonics for these material parameters all asymptote to \( \bar{\nu}_3 \), the third wave front associated with the inner core. This behavior corresponds to case 5

Case (5): \( p_1 = i\beta_1, p_2 = i\beta_2 q_1 \) and \( q_2 \) real or complex conjugates, as

\[
k_h \to \infty |p_1| \to |p_2| a \quad \beta^2 \to \bar{\nu}_3^2
\]

\[
\bar{\nu}_3^2 = \frac{(\bar{\nu}_2 - \bar{\alpha}_2)(\bar{\nu}_1 \bar{v}_2 - \bar{\alpha}_1 \bar{\alpha}_2) - \beta^2(\bar{\nu}_2 + \bar{\alpha}_2) + 2\beta \sqrt{\bar{\alpha}_2} \bar{v}_2 T_C}{(\bar{\nu}_2 - \bar{\alpha}_2)^2} \quad (4.4)
\]

Where \( T_C = \beta^2 - (\bar{\nu}_1 - \bar{\alpha}_1)(\bar{\nu}_2 - \bar{\alpha}_2) \).
A final case, analogous to case 5, is characterized by

\[ q_1 = i\hat{q}_1 q_2 = i\hat{q}_2, p_1 \text{ and } p_2 \text{ real or complex conjugates}, \]

as \( kh \to \infty \left| q_1 \right| \to \left| q_2 \right| \alpha \beta^2 \to \beta^2 \)

\[
v_3^2 = \frac{(y_2 - \alpha_2)(y_1y_2 - \alpha_1\alpha_2) - \beta^2(y_2 + \alpha_2) + 2\beta\sqrt{\alpha_2 y_2 I_c}}{(y_2 - \alpha_2)^2} \quad (4.5)
\]

5. ASYMPTOTIC ANALYSIS

The asymptotic behavior of the dispersion relation is now investigated.

5.1 Surface and interfacial waves

The dispersion relation (3.4) in this case when \( kh \), \( k \to \infty \) tends to

\[
\{q_1 f(q_2)h(q_1) - q_2 f(q_1)h(q_2)\}(-\delta_2^{(\infty)} + \delta_3^{(\infty)} + \delta_4^{(\infty)} - \delta_5^{(\infty)}) = 0,
\]

5.2 Short wavelength limit of the harmonics (\( k \to \infty \))

5.2.1 Case 1: \( v \to v_1 \) with \( \alpha_1 < y_1 \) and \( y_2(y_1 - \alpha_1) - \beta^2 > 0 \)

The first case we consider is when the limiting wave speed of the harmonics is a longitudinal wave associated with the outer layers. Numerical calculations in this case indicate that for all harmonics \( \beta^2 \) approaches \( \alpha_1 \) from above and therefore, from equation (2.3) it is clear that within the region \( \alpha_1 < v < y_1 \) only one of \( q_1 \) and \( q_2 \) is real, the other being purely imaginary, with \( p_1 \) and \( p_2 \) either real or complex conjugates within \( \alpha_1 < v < \beta_2 \). It is assumed, without loss of generality, that \( q_1 = i\hat{q}_1 \), where \( \hat{q}_1 \geq 0 \) is real, and \( \hat{q}_1 \to 0 \) as \( kh, k \to \infty \). Accordingly we seek to expand the dispersion relation (3.4) around the small order quantity \( \hat{q}_1 \). Using equation (2.3) an expansion for the phase speed is obtained, thus
\[ p_2 = \alpha_1 + \left( \gamma_2 + \frac{\beta^2}{\alpha_1 - \gamma_1} \right) \tilde{q}_1^2 + O(\tilde{q}_1^4). \]  

(5.2)

Similar expansions for \( q_2, p_1 \) and \( p_2 \) are then obtained using equation (5.2) with the appropriate form of equation (2.3), namely

\[ q_2 = \tilde{q}_2 + O(\tilde{q}_1^2), \quad p_1 = \tilde{p}_1 + O(\tilde{q}_1^2), \quad p_2 = \tilde{p}_2 + O(\tilde{q}_1^2). \]  

(5.3)

Within which \( \tilde{q}_2, \tilde{p}_1 \) and \( \tilde{p}_2 \) are order 1 quantities defined by

\[ \tilde{q}_2 = \sqrt{\frac{\gamma_2(\gamma_1 - \alpha_1) - \beta^2}{\alpha_2 \gamma_2}} \tilde{p}_1 \cdot \tilde{q}(\alpha_1), \]  

(5.4)

Where

\[ 2\tilde{a}_2 \tilde{p}_2 \delta(\nu^2) = (\tilde{a}_2 (\tilde{a}_1 - \nu^2) + \tilde{p}_2 (\tilde{p}_1 - \nu^2) - \tilde{a}_1^2) \pm \]  

\[ \sqrt{\tilde{a}_2 (\nu^2 - \alpha_1) + \tilde{p}_2 (\nu^2 - \tilde{p}_1) + \tilde{a}_1^2} \]  

\[ - 4\tilde{a}_2 \tilde{p}_2 (\nu^2 - \tilde{a}_1)(\nu^2 - \tilde{p}_1) \]  

(5.5)

The appropriate form of the dispersion relation for large wave number is now obtained by making use of (5.2)-(5.5) in (3.4), thus

\[ \tan(k\tilde{q}_1 h) \{(\tilde{q}_2 h(q_2)f(q_1)\zeta_1) - \tilde{q}_1^2(h(q_1)f(q_2)\tau_1)\} + O(\tilde{q}_1^2) \]  

\[ = \tilde{q}_1^2 \{((\gamma_2 - \sigma_2)(\alpha_1 - \gamma_1)\tilde{g}(\tilde{p}_1) - \alpha_1 \gamma_1 f(\tilde{p}_1))\tilde{h}(\tilde{p}_2) - \tilde{h}(\tilde{q}_2)\} \]  

(5.6)

Within which \( \zeta_1 \) and \( \tau_1 \) are order 1 quantities defined as

\[ \zeta_1 = \tilde{q}_2 \tilde{p}_2 \{((\gamma_2 - \sigma_2)(\alpha_1 - \gamma_1)\tilde{g}(\tilde{p}_1) - \alpha_1 \gamma_1 f(\tilde{p}_1))\tilde{h}(\tilde{p}_2) - \tilde{h}(\tilde{q}_2)\} \]  

\[ - \tilde{q}_2 \tilde{p}_2 \{((\gamma_2 - \sigma_2)(\alpha_1 - \gamma_1)\tilde{g}(\tilde{p}_2) - \alpha_1 \gamma_1 f(\tilde{p}_2))\tilde{h}(\tilde{p}_1) - \tilde{h}(\tilde{q}_2)\} \]  

\[ \tilde{p}_1 \tilde{p}_2 \beta \{((\gamma_2 - \sigma_2)(\alpha_1 - \gamma_1)\tilde{g}(\tilde{p}_2) - \alpha_1 \gamma_1 f(\tilde{p}_2))\tilde{h}(\tilde{p}_1) - \tilde{h}(\tilde{p}_2)\} \]  

(5.7)
\( \tau_1 = \tilde{p}_1 \left\{ (f(\tilde{c}_2) \tilde{g}(\tilde{p}_2) - g(\tilde{c}_2) \tilde{f}(\tilde{p}_2))(\beta \tilde{h}(\tilde{p}_1) - \beta ((\alpha_2 (\alpha_1 - \gamma_1) - \\
alpha_1 \beta))) - \tilde{p}_2 \left\{ (f(\tilde{c}_2) \tilde{g}(\tilde{p}_1) - g(\tilde{c}_2) \tilde{f}(\tilde{p}_1))(\beta \tilde{h}(\tilde{p}_2) - \beta ((\alpha_2 (\alpha_1 - \gamma_1) - \\
alpha_1 \beta))) - \tilde{c}_2 \beta \left\{ \tilde{f}(\tilde{p}_1) \tilde{g}(\tilde{p}_2) - \tilde{f}(\tilde{p}_2) \tilde{g}(\tilde{p}_1) \right\} ((\alpha_2 (\alpha_1 - \gamma_1) - \alpha_1 \beta) - \\
h(\tilde{c}_2)) \right\} \right\} \right\}. \tag{5.8} \\

From equation (5.6) it is deduced that the leading order terms are \( O(1) \)
\( \tan(\kappa \tilde{c}_1 h) \sim G(\tilde{d}_1) \), implying that in the limit \( \tan(\kappa \tilde{c}_1 h) \to 0 \) as \( \tilde{d}_1 \to 0 \),
and therefore
\[ \tilde{d}_1 = \frac{n}{kh} + O(\kappa h)^{-2}. \tag{5.9} \]

Inserting equation (5.9) into (5.2) yields the second order approximation to
the phase speed of the \( n^{th} \) harmonic
\[ \nu_\kappa^2 = \nu_1 + \left( \gamma_2 + \frac{\beta^2}{\alpha_1 - \gamma_1} \right) \left( \frac{n}{kh} \right)^2 + \ldots, \quad n = 1, 2, 3, \tag{5.10} \]

A higher order expansion for the phase speed is obtained by setting
\[ \tilde{d}_1 = \frac{n}{kh} + \frac{\phi_1}{(kh)^2} + O(\kappa h)^{-3}, \quad \tan(\kappa \tilde{d}_1 h) = \frac{\phi_1}{kh} + O(\kappa h)^{-2}, \tag{5.11} \]

Where \( \phi_1 \) is to be determined. If these two expansions are inserted into
equation (3.9) and like powers of \( kh \) equated it is found that
\[ \phi_1 = \left( \frac{(\alpha_2 (\alpha_1 - \gamma_1) - \alpha_1 \beta) f(q_2)}{q_2 h (q_2 (\gamma_2 - \sigma_2) ((\alpha_1 - \gamma_1)) + \xi_1) \xi_1} \right) \times \pi n. \tag{5.12} \]

On inserting equation (5.12) into equation (5.11), and on making use of
equation (5.2), it may be shown that
\[ \nu_\kappa^2 = \nu_1 + \left( \gamma_1 + \frac{\beta^2}{\alpha_1 - \gamma_1} \right) \times \left( \frac{n}{kh} \right)^2 \left( 1 + \frac{1}{kh} \left( \frac{(\alpha_2 (\alpha_1 - \gamma_1) - \alpha_1 \beta) f(q_2)}{q_2 h (q_2 (\gamma_2 - \sigma_2) ((\alpha_1 - \gamma_1)) + \xi_1) \xi_1} \right)^{\frac{1}{n}} \right) + \ldots, \quad n = 1, 2, 3, \ldots \tag{5.13} \]
A comparison of the third order expansions with numerical solutions are shown in Fig. 4. Good agreement with the numerical solutions is observed over a large wave number regime.

Figure 4: comparison of numerical solutions with third order asymptotic expansions for (5.13) for case 1.

5.2.2 Case 2: $\bar{v} \rightarrow \bar{v}_1$ with $\alpha_1 < \bar{p}_1$ and $\bar{p}_2(\bar{p}_1 - \bar{\alpha}_1) - \beta^2 > 0$

Using the same analysis we obtain

$$\beta \bar{v}^2 = \bar{\alpha}_1 + \left( \bar{p}_2 + \frac{\bar{\beta}^2}{\bar{\alpha}_1 - \bar{p}_1} \right) \left( n + \frac{1}{2} \right)^2 \left( \frac{n}{k} \right)^2 \left( 1 + \frac{2\eta_2}{\zeta_2 k} \right) + \cdots, \; n = 1, 2, 3, \ldots$$

(5.14)

Where $\zeta_2$ and $\eta_2$ are order 1 quantities defined as

$$\zeta_2 = \bar{q}_1 \left\{ f(q_2)g(p_2) - g(q_2)f(p_2) \right\} - p_2 \left\{ f(q_1)g(q_2) - h(q_1) \right\} - 121.$$
\[ f(q_2) g(q_2) \left[ h(p_1) - \bar{h}(p_2) \right] - \bar{a}_2 \{ (f(q_1) \bar{q}_1(p_1) -
\]
\[ g(q_1) f(p_2) \right) (\beta h(p_1) - \bar{\beta} h(q_2)) \}, \tag{5.15} \]

\[ r_{i2} = \bar{a}_1 \bar{p}_2 \left\{ (f(q_2) g(p_1) - g(q_2) f(p_1)) \left( \beta h(p_2) - \bar{\beta} h(q_1) \right) \right\} 
\]
\[ - \bar{a}_2 \bar{p}_2 \left\{ (f(q_1) g(p_1) - g(q_1) f(p_1)) \left( \beta \bar{h}(p_2) - \bar{\beta} \bar{h}(q_2) \right) \right\} \]

The form of \( \bar{q}_1, \bar{q}_2 \) and \( \bar{p}_2 \) may be inferred from equation (5.5) by interchanging the material parameters of the outer layers and the inner core.

5.2.3 Case 3: \( \nu \to \nu_2 \) with \( \alpha_1 < \gamma_1 \) and \( \alpha_2 (\alpha_1 - \gamma_1) - \beta^2 > 0 \)

By the same method previously used, we obtain

\[ \rho v_2^2 \gamma_1 + \left( \frac{\beta^2}{\alpha_1 - \gamma_1} \right) \times \left( \frac{n}{k h} \right)^2 \left( 1 + \frac{2}{k h} \left( \frac{u_2 (\alpha_1 - \gamma_1) - u_1 \beta \bar{q}_2}{\bar{q}_2 \gamma_2 - \sigma_2 (\alpha_1 - \gamma_1) + \bar{\gamma}_2} \right) \right) \]
\[ \cdots, \quad n = 1, 2, 3, \ldots \tag{5.16} \]

Fig 5: Comparison of numerical solutions with third order asymptotic expansions for case 4.

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Where \( \zeta_3 \) and \( \eta_3 \) may be readily inferred by replacing the definitions of \( \varphi_1, \varphi_2 \) and \( \varphi_2 \).

5.2.4 Case 4: \( \nu \rightarrow \tilde{v}_2 \) with \( \tilde{\alpha}_1 > \tilde{\varphi}_1 \) and \( \tilde{\alpha}_2 (\tilde{\alpha}_1 - \tilde{\varphi}_1) - \tilde{\beta}^2 > 0 \)

Using the same analysis we obtain

\[
\begin{align*}
\tilde{\mu}^2 = \tilde{\varphi}_1 + \left( \tilde{\alpha}_2 - \frac{\tilde{\beta}^2}{\tilde{\varphi}_1} \right) (n + \frac{1}{2})^2 \left( \frac{n}{k} \right)^2 \left( 1 + \frac{2\eta_4}{\zeta_4 k} \right) + \ldots, \quad n = 1,2,3,\ldots
\end{align*}
\]

(5.17)

Where \( \zeta_4 \) and \( \eta_4 \) are order 1 inferred from definitions of \( \zeta_1 \) and \( \eta_1 \).

5.2.5 Case 5: \( \nu \rightarrow \tilde{v}_3 \) with \( \tilde{\alpha}_1 > \tilde{\varphi}_1 \) and \( \tilde{\alpha}_2 (\tilde{\alpha}_1 - \tilde{\varphi}_1) - \tilde{\beta}^2 > 0 \)

In this case we know numerically that as \( kh, k \rightarrow \infty \) both \( p_1 \) and \( p_2 \) are imaginary and that \( |p_1| \rightarrow |p_2| \), whilst \( q_1 \) and \( q_2 \) are either real or complex conjugates. The limit \( kh, k \rightarrow \infty \) may therefore be examined in this case by setting

\[
\begin{align*}
\tilde{p}_1^2 &= -\tilde{\alpha} + \tilde{\beta}, \quad \tilde{p}_2^2 = -\tilde{\alpha} - \tilde{\beta}, \quad \tilde{\alpha} > 0, \quad \tilde{\beta} \geq 0
\end{align*}
\]

(5.18)

Where \( \tilde{\alpha} \) and \( \tilde{\beta} \) are real and \( \tilde{\beta} \rightarrow 0 \) as \( kh, k \rightarrow \infty \). The values of \( \tilde{\alpha} \) and \( \tilde{\beta} \) may be obtained explicitly from the appropriate form of equation (2.3), thus

\[
\tilde{\alpha} = \frac{\tilde{\alpha}_2 (p_2 \tilde{\varphi}_2 - \tilde{\varphi}_1) + \tilde{\beta}_2 (p_2 \tilde{\varphi}_2 - \tilde{\varphi}_1) + \tilde{\beta}^2}{2\tilde{\alpha}_2 \tilde{\varphi}_2}
\]

(5.19)
By making use of equation (5.19, in conjunction with (2.5), it is possible to obtain the following quadratic for $a$:

$$
\tilde{E} = \sqrt{\frac{\tilde{a}_2 \left( \tilde{\beta} \nu^2 - \tilde{a}_1 \right) + \tilde{\nu}_2 \left( \tilde{\beta} \nu^2 - \tilde{\nu}_1 \right) + \tilde{\beta}^2 - 4 \tilde{a}_2 \tilde{\nu}_2 \left( \tilde{\beta} \nu^2 - \tilde{\nu}_1 \right)}{2 \tilde{a}_2 \tilde{\nu}_2}}
$$

which for small $\tilde{E}$ implies that

$$
\tilde{a} = \tilde{a}_c + \tilde{a}_1 \tilde{E}^2 + O(\tilde{E}^4) \quad (5.21)
$$

Where

$$
\tilde{a}_c = \frac{(\tilde{a}_2 + \tilde{\nu}_2) \sqrt{\tilde{\beta}^2 \tilde{\zeta} - \tilde{\zeta} - \tilde{\beta}^2}}{(\tilde{a}_2 - \tilde{\nu}_2)^2} \quad (5.22)
$$

Within which

$$
\tilde{\zeta} = \tilde{\beta}^2 - (\tilde{a}_1 - \tilde{\nu}_1)(\tilde{a}_2 - \tilde{\nu}_2), \quad (5.23)
$$

On inserting equation (5.18) and equation (5.21) into the dispersion relation (3.4) and expanding for small $\tilde{E}$ it is deduced for high wave number and small $\tilde{E}$ that either
\[
\bar{q}_1 h(\bar{q}_1) f(\bar{q}_2) - \bar{q}_2 h(\bar{q}_2) f(\bar{q}_1) + O(\bar{E}^2) = 0, \quad (5.24)
\]

Or
\[
(\bar{q}_1 \Gamma_3 + \bar{q}_2 \Gamma_2) \bar{E} S(\bar{a}_0) + \Gamma_0 \bar{E} C(\bar{a}_0) = \Gamma_0 \bar{E} C(\bar{E}) - (\bar{q}_2 \Gamma_2 + \bar{q}_1 \Gamma_4) S(\bar{E}) + O(\bar{E}^2)
\]
(5.25)

Within which \( \Gamma_m(m = 1, \ldots, 6) \) are order one terms, \( S(\bar{a}_C) \), \( C(\bar{a}_C) \), \( S(\bar{E}) \) and \( C(\bar{E}) \) are trigonometric terms defined as
\[
C(\bar{a}_C) = \cos 2\sqrt{\bar{a}_C} \bar{E}, \quad S(\bar{a}_C) = \sin 2\sqrt{\bar{a}_C} \bar{E},
\]
\[
C(\bar{E}) = \cos \left( \frac{\bar{E} \bar{a}_C}{\sqrt{\bar{a}_C}} \right) \quad S(\bar{E}) = \sin \left( \frac{\bar{E} \bar{a}_C}{\sqrt{\bar{a}_C}} \right)
\]

And \( \bar{a}_1 \) and \( \bar{a}_2 \) are order 1 term which may be found by setting \( \nu = \bar{v}_2 \) in equation (2.3) is again the appropriately specialized form of the Rayleigh surface wave equation evaluated when \( \nu = \bar{v}_2 \). From equation (5.25) it is readily deduced that \( C(\bar{E}) \sim c(\bar{E}) \), implying \( S(\bar{E}) \rightarrow 0 \) as \( \bar{E} \rightarrow \infty \), and therefore
\[
\bar{E} = n \frac{\sqrt{\bar{a}_0}}{\bar{p}} + O(\bar{E}^2) \quad (5.26)
\]

A second order expansion for the phase speed for the \( \eta^{th} \) harmonic may now be obtained on inserting equation (5.26) into (5.19)_1 and (5.20), to obtain
\[
\beta \nu_{\eta}^2 = \frac{1}{\bar{a}_2 + \bar{p}_2} \left\{ \bar{a}_1 \bar{a}_2 + \bar{p}_1 \bar{p}_2 - \beta^2 + 2 \bar{a}_2 \bar{p}_2 \bar{a}_C (1 + \bar{a}_1 \left( \frac{\eta}{\bar{p}} \right)^2) \right\} + \ldots, \quad n = 1, 2, 3, \ldots \quad (5.27)
\]
A higher order expansion for the phase speed may be found by setting

\[ \tilde{c} = n \sqrt{\frac{a_c}{k}} + \frac{\varphi_5}{(k)^2} + \mathcal{O}(k)^{-2}, \]  

(5.28)

From which we infer that

\[ \sin \left( \frac{\tilde{c} k}{\sqrt{\alpha}} \right) = (-1)^n \frac{\varphi_5}{\sqrt{\alpha} k} + \mathcal{O}(k)^{-3}, \quad \cos \left( \frac{\tilde{c} k}{\sqrt{\alpha}} \right) = (-1)^n + \mathcal{O}(k)^{-2} \]  

(5.29)

Where \( \varphi_5 \) is to be determined. On inserting equation (5.28) and (5.29) into equation (5.27) and comparing like powers of \( k \) it is obtained that

\[ \varphi_5 = -n \left\{ \frac{\left\{ (-1)^n \Gamma_2 + S(\alpha) \Gamma_3 + \Gamma_2 \Gamma_5 \right\} + \mathcal{O}(\Gamma_1)}{(-1)^n \left\{ \Gamma_1 \right\}} \right\}, \]  

(5.30)

Fig. 6: comparison of numerical solutions with third order asymptotic expansions for (5.31) for case 5. The same material parameters as in Fig.3 have been used.

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Equation (5.30) can now be used in conjunction with equations (5.19), (5.21) and (5.28), to obtain

\[ \hat{\beta} \nu_n^2 = \frac{1}{\tilde{a}_2 + \tilde{\gamma}_2} \left\{ \tilde{a}_1 \tilde{a}_2 + \tilde{\gamma}_1 \tilde{\gamma}_2 - \hat{\beta}^2 + 2\tilde{a}_2 \tilde{\gamma}_2 (\tilde{a}_c + \tilde{a}_c \tilde{a}_1 (\frac{n}{k})^2) \left( 1 + \frac{2\tilde{a}_\infty}{\sqrt{\tilde{a}_c k}} \right) \right\} + \cdots, n = 1, 2, 3, \ldots \] (5.31)

where \( \tilde{\varphi}_n = \frac{\varphi_n}{n} \).

Figure (6) shows that once the asymptotic expansions are taken to third order, and the trigonometric terms include, better agreement with the numerical solutions is obtained.

**5.2.6 Case 6: \( \nu \to \nu_3 \) with \( \alpha_1 > \gamma_1 \) and \( \alpha_2 (\alpha_1 - \gamma_1) - \beta^2 < 0 \)**

The final case to consider in the high wave number limit arises when both \( q_1 \) and \( q_2 \) are imaginary and \( |q_1| \to |q_2| \) as \( kh, k \to \infty \), whilst \( p_1 \) and \( p_2 \) are either both real or complex conjugates. The limit \( kh, k \to \infty \) is therefore examined by setting

\[ q_1^2 = -a + b, \quad q_2^2 = -a - b, \quad a > 0, \quad b \geq 0, \] (5.32)

Where \( a \) and \( b \) are real and \( b \to 0 \) as \( kh, k \to \infty \) which for small \( b \) implies that

\[ a = a_c + a_1 b^2 + O(b^4), \] (5.33)

On making use of expansions shown in equations (5.32) and (5.33) it is deduced that for high wave number equation (3.4) takes the form
\begin{align}
\hat{\alpha_1} \hat{\alpha_2} & \{ h(q_1)f(q_1) \delta_1 + h(q_2)f(q_2) \delta_2 \} \\
& - \{ \hat{\alpha_1} h(q_1)f(q_2)(C(a_c)S(E) \delta_2 + S(a_c)C(E) \delta_5) \\
& - \hat{\alpha_2} h(q_2)f(q_1)S(a_c)C(E) \delta_2 - C(a_c)S(E) \delta_5 \} \\
& + \{ \hat{\alpha_1} h(q_2)f(q_2)C(a_c)S(E) \delta_3 + \hat{\alpha_1} S(a_c)C(E) \delta_4 \} \\
& - \hat{\alpha_2} h(q_2)f(q_1)S(a_c)C(E) \delta_3 + \hat{\alpha_1} C(a_c)S(E) \delta_4 \} \\
& = 0 \quad (5.34)
\end{align}

Within which \( \hat{\alpha_1} \) and \( \hat{\alpha_2} \) are approximated by

\begin{align}
\hat{\alpha_1} & = \sqrt{a_c} - \frac{b}{2} \sqrt{a_c} + \xi b^2 + O(b^3) \\
\hat{\alpha_2} & = \sqrt{a_c} + \frac{b}{2} \sqrt{a_c} + \xi b^2 + O(b^3) \quad (5.35)
\end{align}

And \( \xi = (4a_c a_1 - 1)/8a_c \). It will be found that the leading order term of equation (5.34) vanishes in the limit, thus necessitating the inclusion of \( O(b^2) \) terms in all expansions of the components of the dispersion relation.

By making use of equation (5.19) and (5.33) we have

\begin{align}
\mu \nu_h^2 = \frac{1}{a_2 + \gamma_2} \{ \alpha_1 \alpha_2 + \gamma_1 \gamma_2 - \beta^2 + 2\alpha_2 \gamma_2(a_c + a_1 b^2) \} + O(b^4). \quad (5.36)
\end{align}

By making use of equation (5.36) in conjunction with the appropriate form of equation (2.3), and after a little algebraic manipulation, expansions for \( p_1 \) and \( p_2 \) are obtained, namely

\begin{align}
p_1 = p_1 + p_1 b^2 + O(b^4), \quad p_2 = p_2 + p_2 b^2 + O(b^4), \quad (5.37)
\end{align}
Fig 7: Comparison of numerical solutions with second order asymptotic expansions for case 6.

\[ p_1 = \left( \frac{\lambda_1 + \mu_c}{2 \varphi_2 \delta_2} \right)^{\frac{1}{2}}, \quad p_1 = \frac{\lambda_1 + \mu_2}{(2 \varphi_2 \delta_2)^{\frac{1}{2}} (\lambda_1 + \mu_c)^{\frac{1}{2}}}, \]

\[ \nu_2 = \left( \frac{\lambda_1 - \mu_c}{2 \varphi_2 \delta_2} \right)^{\frac{1}{2}}, \quad \nu_2 = \frac{\lambda_1 - \mu_2}{(2 \varphi_2 \delta_2)^{\frac{1}{2}} (\lambda_1 - \mu_c)^{\frac{1}{2}}}, \]

And within which

\[ \lambda_1 = \bar{a}_2 (\bar{a}_1 - \alpha_1) - \bar{p}_2 (\alpha_1 - \bar{p}_1) - \beta^2, \]
\[ \lambda_1 = -\alpha_2 (\bar{a}_2 + \bar{p}_2), \]
\[ \alpha_1 = \frac{\alpha_1 \alpha_2 + \gamma_1 \gamma_2 - \beta^2 + 2 \alpha_2 \gamma_2 \alpha_c}{\alpha_2 + \gamma_2}, \]
\[ \alpha_2 = \frac{2 \alpha_2 \gamma_2 \alpha_1}{\alpha_2 + \gamma_2}. \]
\[ \mu_c = (\beta^4 + 2\beta^2(\alpha_2 - \alpha_1 + \varphi_2(\alpha_1 - \varphi_1) + (\alpha_2 - \alpha_1)^2) + 2\varphi_2(\alpha_1 - \varphi_1))^2, \]

\[ \mu_z = (2\alpha_2\beta^2(\alpha_2 + \varphi_2) - 2\alpha_2(\alpha_2 - \varphi_2)\varphi_2(\alpha_1 - \varphi_1) - \alpha_2(\alpha_1 - \alpha_1)) / 2\mu_c. \]

On making use of equations (5.35)-(5.37), in conjunction with equation (5.34) the dispersion relation in the high wave number region may be cast in the relatively simple form

\[ A_1 + (A_2 + C(a_0)A_4 - S(a_0)A_5)\ell^2 = C(\ell)(A_4 + A_4^2) - S(\ell)A_5 + O(\ell^2) \] (5.38)

Where \( C(a_c), S(a_c), C(\ell) \) and \( S(\ell) \) are trigonometric terms which may be inferred from the definitions given after equation (5.25), and \( A_m \) are order 1 quantities. It is readily deduced from equation (5.38) that the leading order is

\[ \left( 1 - \cos \frac{b}{\sqrt{a_c}} \right) \approx C(\ell) \sin \frac{b}{\sqrt{a_c}}, \] (5.39)

Thus implying that \( \cos \frac{b}{\sqrt{a_c}} \to 1 \) as \( kh, k \to \infty \) and therefore

\[ \ell = 2\sqrt{a_c} \frac{n}{kh} + O(kh)^{-2}. \] (5.40)

A second order approximation to the phase speed is obtained by inserting equation (5.40) into equation (5.36), to find

\[ \nu \approx \frac{1}{a_2 + \gamma_2} \left\{ \alpha_1 \alpha_2 + \gamma_1 \gamma_2 - \beta^2 + 2\alpha_2 \gamma_2 a_0 \{ 1 + 4a_1 \frac{n}{k} \}^2 \right\}. \] (5.41)
We then seek a higher order expansion by setting
\[
E = 2\sqrt{a_c} \frac{n}{kh} + \frac{\phi_{\epsilon}}{(kh)^2} + O(kh)^{-3}.
\] (5.42)

From which it is deduced that
\[
\sin \left( \frac{b}{\sqrt{a_c}} \right) = \frac{\phi_{\epsilon}}{\sqrt{a_c} kh} + O(kh)^{-2}, \quad \cos \left( \frac{b}{\sqrt{a_c}} \right) = 1 - \frac{\phi_{\epsilon}^2}{2a_c (kh)^2} + O(kh)^{-4} \] (5.43)

Fig 8: Comparison of numerical solutions with third order asymptotic expansions for case 6.

Where \( \phi_{\epsilon} \) is to be determined and it is noted that it is necessary to include \( O(kh)^{-4} \) term in the expansion (5.43) due to the vanishing of the leading order term prior to the derivation of equation (5.38). Inserting equation (5.42) and (5.43) into equation (5.38) and comparing like powers of \( kh \) reveals that the equation is identically zero at leading order, the next order yielding the following quadratic equation for \( \phi_{\epsilon} \)
A quadratic equation for $\phi_{e}$ is obtained in this case, whilst in the previous five cases a single value for $\phi_1, \ldots, \phi_5$ was obtained. However, it has been verified numerically that for particular values of $n$ the two solutions indicated in equation (5.44) corresponds to two distinct branches of the dispersion relation. This is not too surprising as we have seen in figure 7 that the harmonics group together to form distinct pairs in the high wave number region. Indeed the second order approximation to the phase speed in equation (5.41) gives asymptotic solutions which pass between adjacent pairs of harmonics, thus in order to obtain accurate asymptotic solutions the expansion must be taken to at least third order. Equation (5.44) may therefore be used, in conjunction with equations (5.41) and (5.36) to obtain

$$\frac{A_1}{2a_0} \phi_{e}^2 + 2n A_0 \phi_{e} + 4a_c(n)^2 \{A_2 - A_4 + A_5 C(a_c) - A_6 S(a_c)\} = 0. \quad (5.44)$$

Where $\hat{\phi}_{e} = \phi_{e} \hat{n}, \hat{\phi}^{+}$ and $\hat{\phi}^{-}$ representing solutions of equation (5.44) associated with the positive and the negative square root associated with the discriminant, respectively. Figure 7 shows the second order expansions obtained in equation (5.41). Each value on $n$ used in (5.41) generates an approximate solutions that passes between the $2n$th and $(2n - 1)$th branches of the numerical dispersion curves, clearly indicating that the second order expansion is not sufficient to describe the behaviour of the numerical solutions accurately. When the expansions are taken to higher order, as in
figure (8), the oscillations are fully accounted for by the trigonometric functions within the third order term of equation (5.45) and the two roots of the quadratic equation in $\phi$ now provide an asymptotic approximation for each harmonic.

**CONCLUDING REMARKS**

In this paper, the dispersion of anti-symmetric waves in a symmetric three-layres plate has been investigated. The investigation has been carried out within the framework of compressible, pre-stressed elastic media, with each layered assumed composed of such material. The dispersion relation is a mathematically elaborate transcendental equation, derived from a six by six determinant, which gives phase speed as an implicit function of wave number. A through numerical analysis reveals the nature of the dispersion relation and provides a sign post for the derivation of some short wave approximation of the harmonics in each of the six different cases which are shown to exist. These approximations provide an explicit relationship between phase speed and wave number and shown to provide highly accurate approximations over a wide wave number range.

**REFERENCES**


